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ABSTRACT

This paper discusses recent research on cognition and its implicaticns for mathematics education. Learning as a generative process, structural organization, processing of information, processes and structures, individual differences, brain research, higher-order processes, motivation, and delay-retention effects are discussed. (MS)

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MATHEMATICS EDUCATION REPORTS

RECENT RESEARCH IN COGNITION APPLIED TO MATHEMATICS LEARNING

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ERIC Information Analysis Center for
Science, Mathematics, and Environmental Education
The Ohio State University
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Mathematics Education Reports are being developed to disseminate information concerning mathematics education documents analysed at the ERIC Information Analysis Center for Science, Mathematics, and Environmental Education. These reports fall into three broad categories. Research reviews summarize and analyze recent research in specific areas of mathematics education. Resource guides identify and analyze materials and references for use by mathematics teachers at all levels. Special bibliographies announce the availability of documents and review the literature in selected interest areas of mathematics education. Reports in each of these categories may also be targeted for specific sub-populations of the mathematics education community. Priorities for the development of future Mathematics Education Reports are established by the advisory board of the Center, in cooperation with the National Council of Teachers of Mathematics, the Special Interest Group for Research in Mathematics Education of the American Educational Research Association, the Conference Board of the Mathematical Sciences, and other professional groups in mathematics education. Individual comments on past Reports and suggestions for future Reports are always welcomed by the editor.

Preface

This paper, which discusses some recent research on cognition and its meaning for mathematics education, was commissioned by the ERIC Information Analysis Center for Science, Mathematics, and Environmental Education. The paper was presented by Professor Wittrock at a meeting of the Special Interest Group for Research in Mathematics Education at the annual convention of the American Educational Research Association on February 28, 1973. It is with great pleasure that we now make this paper available to the wider mathematics education community as a Mathematics Education Report.

Marilyn N. Suydam
Editor

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RECENT RESEARCH IN COGNITION APPLIED TO MATHEMATICS LEARNING

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I. Introduction

An important, welcome evolution is occurring in the study of human learning and retention. As a result of recent research by many people, we are increasing our understanding of the processes involved in cognition and in the learning of mathematics. I want to discuss some of this recent research and its meaning for mathematics education.

As a way to organize this research and to focus your interests upon its meaning for mathematics education, I will begin with a hypothesis about human learning that I have developed from my research in cognition, discovery learning, and instruction in schools. I will then present a sample of the empirical studies which led to the generation of the hypothesis. Last of all, I will discuss some of the meaning of this research for mathematics learning.

The first point to emerge will be that we can be proud of the research in the learning of mathematics, including the development of curricular materials. The second point will be a recommendation that research in mathematics learning should increasingly be devoted to studying the step-by-step specific and higher-order intellectual processes which students engage in when they learn mathematics; such as when they are adding, subtracting, differentiating, and integrating.

The hypothesis and empirical studies I will present focus upon the cognitive, generative processes that are involved in the learning of mathematics. These processes could equally well be presented in simpler S-R terminology. I prefer the cognitive model because I want to emphasize the learner's step-by-step processing of information.

The data to be discussed will probably arouse in you recollections of the dilemmas about nativism and the role of abstractions in memory, which Plato and Aristotle wrote about. In the Meno, Plato taught the slave boy to prove that the diagonal of a square is equal to the side of a square twice the area of the given square. To Plato, it seemed that the abstractions were inherited and primary.

Aristotle circumvented Plato's stress upon nativism and the primacy of abstractions by reducing the abstractions to nothing more than commonalities across particulars. He made sensory data and the particulars the focus of human learning. The hypothesis and data I will present may appear to raise again this ancient dilemma. I will let you decide whether that is the case.

Learning as a generative process

Succinctly, but abstractly stated, the hypothesis is that human learning with understanding is a generative process involving the construction of (1) organizational structures for storing and retrieving information and (2) processes for relating new information to the stored information.

Stated more directly, all learning which involves understanding is discovery learning. We can determine the effects of instruction best in terms of what the instruction causes the learner to do. Effective instruction causes the learner to generate a relationship between new information and previous experience.

Further, I contend that we should not construe learning, even so-called reception learning, as a passive reception of someone else's organizations and abstractions. It is better, I maintain, to look beyond nominal stimuli to the functional stimuli they become for the learner. In the past, we have emphasized the importance of environment and instruction on the learner. New stress must be placed upon the active role of the individual in learning.

From this point of view, there is no one best method of teaching all students, although there may be one best logical organization of the subject matter. It is possible for reception-learning treatments to lead to discovery learning: however, this is contingent on the teacher's ability to build upon the learner's previous knowledge and what the resultant instruction causes the learner to do. A variety of teaching methods will be needed by each student, depending upon his background and its relationship to the subject matter we are teaching him.

In sum, learning with understanding can occur with discovery treatments or with reception treatments. The important point is what these treatments cause the learner to do.

The best way I know to summarize the above conception is as follows. Although a student may not understand sentences spoken to him by his teacher, it is highly likely that a student understands sentences which he generated himself. I believe that in this context, generation and understanding are closely related; possibly one causes the other, or perhaps the terms are synonymous.

Demonstrations of learning as a generative process

Now let me elaborate upon the above argument by presenting you with two brief demonstrations, one perceptual and one verbal. For the first demonstration, please look at the black irregularly shaped objects in Figure 1.

Figure 1.

Look at the black objects above.
What are they?

What do you see? You probably see only black irregularly shaped objects because your previous experience has trained you to treat black as foreground on white pages, and also because you were instructed to process the information as black irregularly shaped objects.

Now try to process the information in Figure 1 differently. Process the black as background and the white as foreground. When you are successful at it, you will see the word PLAY in capital letters. The point here is that the way in which you process structurally-organized information is crucial to the meaning you derive from it. You might reflect on this example the next time one of your students doesn't get the point. He may be generating black objects instead of words.

For a demonstration from the verbal area, read the paragraph in the box (Figure 2) and generate a title for it.

> The procedure is actually quite simple. First you arrange things into different groups depending on their makeup. Of course, one pile may be sufficient depending on how much there is to do. If you have to go somewhere else due to lack of facilities that is the next step, otherwise you are pretty well set. It is important not to overdo any particular endeavor. That is, it is better to do too few things at once than too many. In the short run this may not seem important, but complications from doing too many can easily arise. A mistake can be expensive as well. The manipulation of the appropriate mechanisms should be self-explanatory, and we need not dwell on it here. At first the whole procedure will seem complicated. Soon, however, it will become just another facet of life. It is difficult to foresee any end to the necessity for this task in the immediate future, but then one never can tell. [Bransford and Johnson, 1972.]

Figure 2.

The words seem to apply to many tasks, but identifying a specific title is difficult for most readers. The paragraph seems very vague--

until you are told that the title of the passage is "Washing Clothes." The title provides the cue that allows you to relate the vague terms contained in the passage to your knowledge of washing clothes.

Now we will turn to the research literature on generative structures and processes.

II. Recent Research

The recent research I will present now is divided into two sections: (1) structural organization and (2) processing and coding of information. I will present representative studies only and not attempt to be exhaustive in citing relevant research in the field.

Structural organization

The example given above which dealt with the washing of clothes was taken from a study reported by Bransford and Johnson in the December 1972 issue of the Journal of Verbal Learning and Verbal Behavior. They found (page 723) that giving the title for the story before the story was read greatly enhanced its comprehension and recall, while giving the title after the story did not increase comprehension or retention. It seems that the structural organization in this vague story is too weak to allow much generative processing of its information.

The second research study to be cited is by Bower and Winzenz (1969). They presented students with strings of digits such as the following:

First String (17) (683) (9452) (7) (56)

Second String (176) (8) (394) (5275) (6)

The digits of each string and their order remained the same across the strings, while the grouping changed from the first to the second string. The results showed that altering the organization of the string did not improve learning across altered strings, although the digits and their order of presentation remained the same. Apparently, each string was learned as though it were a completely new one. By changing the structural organization, it seems that a new task is presented to the learner. In fact, we could have predicted proactive interference in this task, based on the similarity of the digits across the strings. Because proactive interference among digits did not occur, it could mean that the groups of digits are treated as the units to be learned. We need to examine our concepts of the units learned and stored. These units may be larger than we have assumed.

The third empirical study on structural organization (Clarks, Doctorow, and Wittrock, in preparation) involves the teaching of reading to public school children. Two hundred twenty-two students were randomly assigned to two conditions which varied the meaningfulness of 20 per cent of the words in a commercially-published story designed to teach reading. From the generative hypothesis presented above, we predicted that the semantic properties of the words would have decided effects upon story comprehension and retention. The

reason for the prediction is related to an "all-or-none" hypothesis. By increasing word meaning, the organization of the narrative should be discovered on an all-or-none basis. As you can see from Table 1, story comprehension and retention were nearly doubled by changing the meaning of 20 per cent of the words in the story from less-meaningful ones (e.g., lad) to more-meaningful ones (e.g., boy). The use of unfamiliar terms can severely retard comprehension of the structural organization of a story. One implication is that a few unfamiliar terms may have a similar decided effect upon mathematics learning.

The fourth study (Doctorow, Wittrock and Marks, in preparation) was designed to test further the generative conception of learning presented above by using a familiar organization (a familiar story) to teach the meaning of new words to 482 elementary school children. Again we reasoned that a familiar structural organization should enable a child to "discover" the meanings of new words without our defining them for him. From Table 2 you can see the results from this research study, using commonly available teaching materials. All children read the same stories twice; however, the high-meaningful group read a familiar version first, one which used highly meaningful words, and then read the same story but with many new unfamiliar synonyms that did not change the meaning of the story. The control group read the version with the low-meaningful words twice. On the test of definitions of words, the group which read the familiar version of the story first did much better than the control group.

Table 1

Means and Standard Deviations of the Percentage of Items Correct on the Posttests of Reading Comprehension and Retention

Reading Level	Test of Comprehension At Subjects' Reading Level		Test of Comprehension Above Subjects' Reading Level		Cloze Retention Test	
	High Frequency Treatment	Low Frequency Treatment	High Frequency Treatment	Low Frequency Treatment	High Frequency Treatment	Low Frequency Treatment
High	73 (14)	48 (19)	67 (14)	47 (17)	58 (15)	36 (16)
Middle	74 (14)	47 (16)	57 (14)	29 (13)	53 (16)	31 (15)
Low	71 (14)	45 (22)	61 (14)	36 (15)	55 (15)	20 (16)
					51 (14)	29 (14)

Table 2

Using Word Meaning to Improve
Reading and Listening Comprehension

Reading Levels	Tests	MEAN SCORES FOR THE FOUR EXPERIMENTAL GROUPS			
		Reading		Listening	
		1) High Freq.	2) Low Freq.	3) High Freq.	4) Low Freq.
Low	Comp. Questions	7.0	4.2	8.1	3.9
	Vocabulary	7.3	5.3	7.8	5.7
	Retention	19.2	8.9	20.2	9.0
Middle	Comp. Questions	6.9	4.0	7.9	4.2
	Vocabulary	10.9	7.0	10.2	7.1
	Retention	45.5	25.2	52.4	24.5
High	Comp. Questions	7.3	4.8	7.8	4.7
	Vocabulary	12.7	10.5	13.5	10.1
	Retention	53.6	26.6	61.2	26.4

We never defined any words for either group. Neither did we associate any of the synonyms with each other. Instead, we introduced a familiar structural organization, a story, and used it as a meaningful context to teach definitions of new words in a relatively painless, efficient way. I believe the idea of a familiar organization can also be used to teach the definitions of unfamiliar mathematical terms.

It is proper to end this section on structural organization by mentioning the research of David Ausubel, who pioneered in this area with his books and articles on meaningful verbal learning. His studies on learning Buddhism as a new organization which can be imposed upon one's previously learned religion continues to stir interest in structural organizations.

From an S-R perspective, Benton Underwood (1957) also fostered work in this area with his classical research on the importance of proactive processes in learning and retention. As I said earlier, either associationistic or cognitive approaches are capable of handling the data of these studies. In fact, to my surprise, when these two approaches were put to an explicit test in a study dealing with the retention of sentences (Anderson and Bower, 1971), the associationistic model was supported. Although the associationists have won in this arena, the controversy which Plato and Aristotle began over the roles of abstractions and particulars in learning has not yet been resolved. Unfortunately, no end to it is in sight.

Processing of information

Under this heading are the research studies in which learners generate groups or other large units such as stories, sentences, and images from the stimuli they have been given. The results of these studies occasionally are dramatic.

For example, Bower and Clark (1969) gave subjects 12 different lists of ten unrelated nouns. One group of subjects was asked to learn each list in whatever manner each person wished, keeping the order of the words the same. The experimental group was asked to make a meaningful story from each list. The results need little elaboration. Statistical tests are not necessary either, because the mean gain from the control to the experimental group was 79 per cent--from 14 per cent to 93 per cent! The generation of a story, a thematic organization, greatly reduced interference among the 12 lists and facilitated the retention of the words and their serial order.

A number of researchers (e.g., Rohwer, 1973; Anderson, 1970; and Bobrow, 1970) have found that processing words into a sentence also facilitates their retention. I will not discuss the facilitating effect of processing words into sentences any further here, because of the apparent similarity of this type of processing to the generation of stories.

The next type of generative processing I wish to discuss is imagery, which Alan Paivio (1971) and others have extensively studied recently.

Britta Bull and I (in preparation) have just completed a study on the teaching of word definitions to 90 fifth-graders, using three learning conditions. The instructions for the groups differed as follows: (1) generate (draw) an image of the word and its definition; (2) trace a picture (image given) representing the word and its definition; or (3) learn the verbal definition by coping it. We found that the group means for retention of the definition ranked from high to low in the order given above. The group that generated their own images remembered the definitions best one week later. There was no statistically significant difference between the other two groups. Again, the generative semantic processing hypothesis fits the data. It does not seem that the mode of generation (story, sentence, or imagery) regularly discriminates among performance. What does seem to be important is that some type of generation be called upon.

Processes and structures

To test the latter possibility, Sheila Goldberg and I (in preparation) combined into one study the above three different types of generative processes commonly studies in the literature: (1) imagery, (2) stories, and (3) sentences, in addition to a control group. We randomly assigned college subjects to these four conditions and gave them lists of words to remember. The words varied in meaningfulness and in imagery value. The results are presented in Figure 3. We found no important mean differences among the three

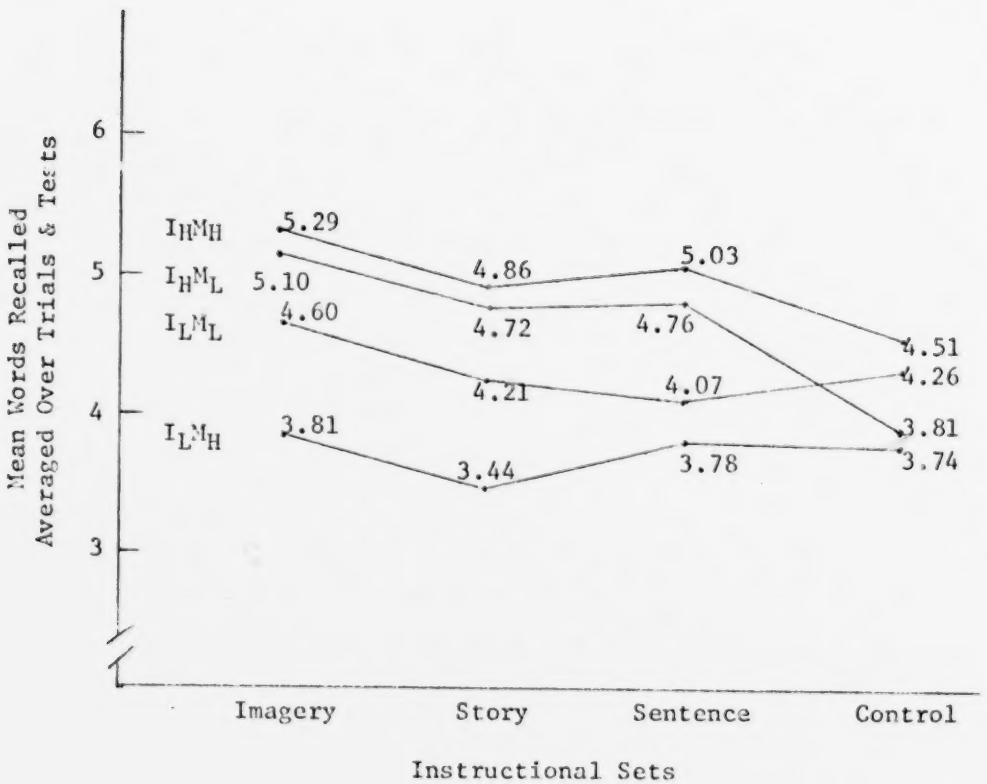

Figure 3. Results from Wittrock and Goldberg study

generative processes, although the control group did less well than the other three groups. However, we did find that the imagery and meaningfulness values of the words used in the study influenced retention. The history of associations to words is an important factor in determining their effect upon retention.

Dr. Goldberg and I have replicated this study with public school children. We found that verbal processing conditions, sentences or stories, do facilitate retention in children, when compared with the imagery conditions. Perhaps, with children, the nonverbal processes need to be primed more than the verbal ones. This issue is still being investigated. The point is that generative processing of information has effects upon retention.

The clearest picture I can present to you of how different organizational structures and different processes of organizing information fit together is the one presented in Figure 4. This figure shows a hierarchical structure which John Carter and I adapted from a study by Bower et al. (1969). Figure 5 presents the randomly-ordered hierarchy for the concept of minerals. If you struggle with it for a while, you will discover that this hierarchy can be rearranged into a properly ordered one, with the word "minerals" at the top. In addition to the random hierarchy, we developed a proper hierarchy as shown in Figure 6. Last, we prepared an unordered hierarchy (see Figure 7) by putting conceptually unrelated words into a hierarchy. These words were compararable in frequency values to the words in the other two hierarchies.

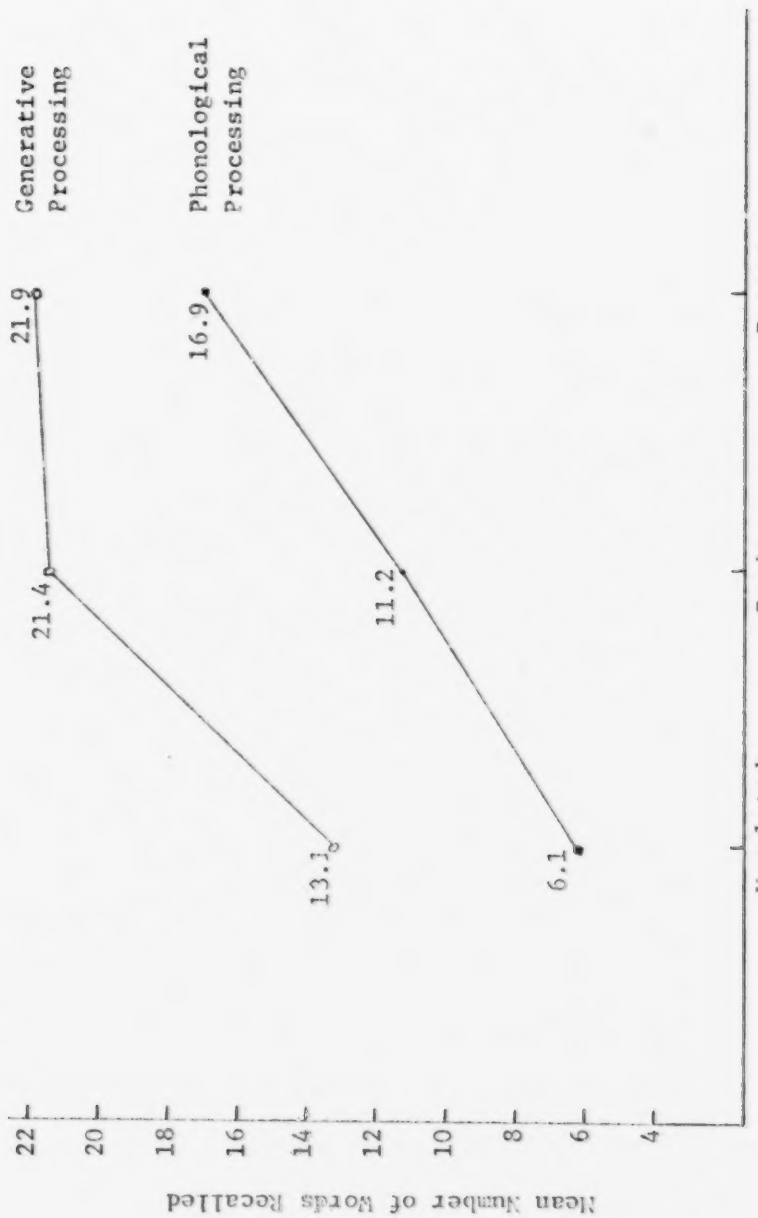

Figure 4. Results from Wittrock and Carter study

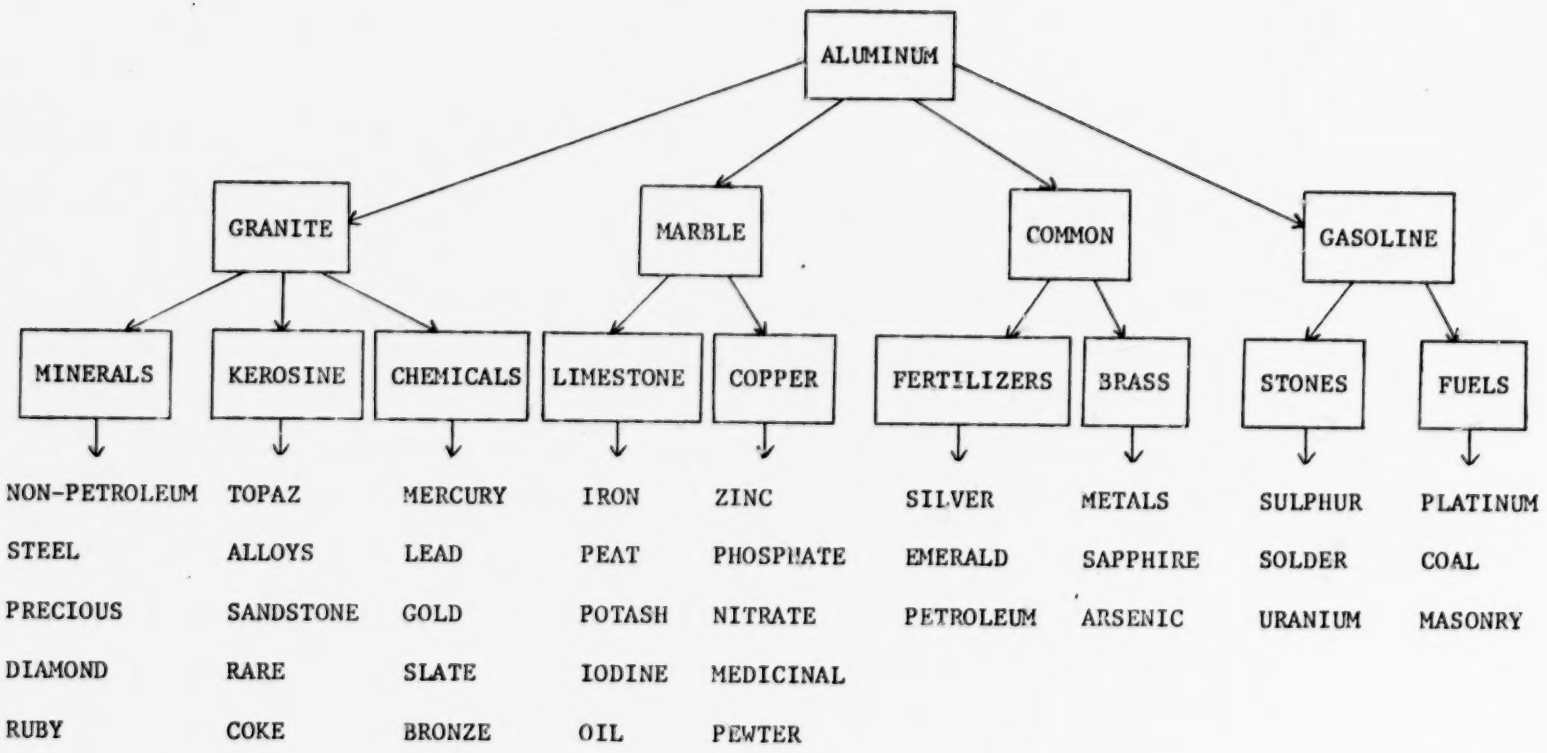

Figure 5. Randomly-ordered hierarchy (Wittrock and Carter study)

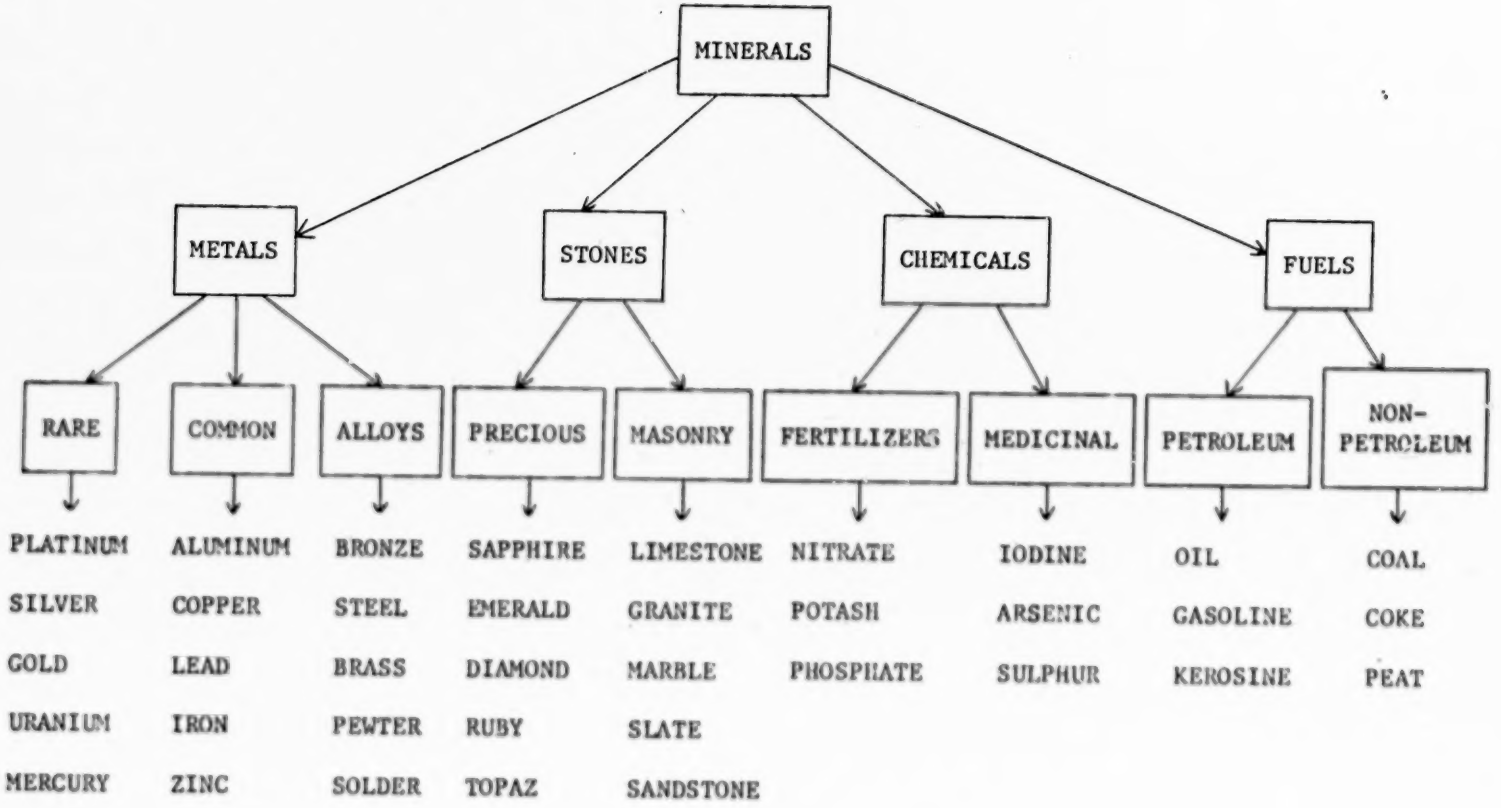

Figure 6. Proper hierarchy (Wittrock and Carter study)

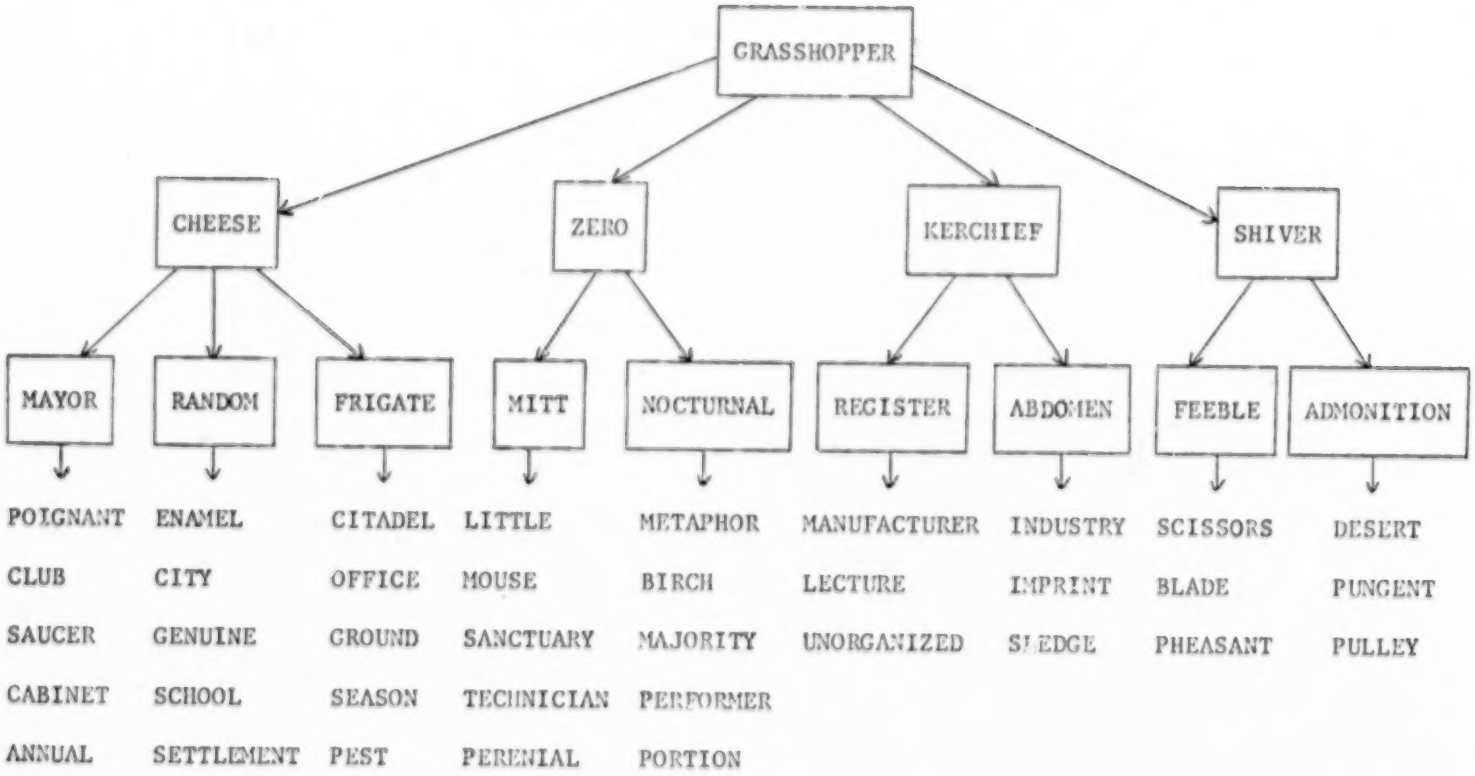

Figure 7. Unrelated hierarchy (Wittrock and Carter study)

We gave these hierarchies to college subjects under two different processing conditions. In the first, or control condition, the learners were asked to copy the hierarchies. The second, or generative processing condition, emphasized rearranging the hierarchies until an order was found that "made sense."

The interesting results are found in Figure 4. With the rote processing indicated in the bottom line, increasing the structural organization increases retention in nearly a linear fashion. However, the generative processing instructions had sizable effects upon retention at all levels. Generative processing had the most marked effect when the organization was in the learners' repertoires, but not made explicit, as in the randomly-ordered hierarchy (Figure 5). Because generative processing facilitated retention at all three levels of structural organization, I believe that it makes sense to separate these two types of factors as I have done in the outline of this paper. This separation helps to understand sometimes conflicting results on discovery learning.

The data in Figure 4 summarize what I believe to be an important relationship between structural (content) organization and generative, semantic processing of information. It was this study which led me to conclude that learning with understanding is a generative process.

The above line of research gives a new understanding of what is involved in meaningful learning. Meaningful learning requires a structural or content organization and enough relevant background

knowledge to enable one to discover the new organization when it is not made explicit. Whether or not the organization of the content is made explicit does not seem to be crucial. It is more important that the learner generatively process any kind of organization he is given. The combination of appropriate background and generative processing of the new information should facilitate his understanding of the organization presented to him.

Individual differences

I have very recently turned my interests to the study of aptitude-treatment-interactions (ATI) and to individual differences in processing variables. The reason that ATI research has not often produced significant results is probably because we have not often chosen to study the relevant processes that the learner engages in when he learns a given subject matter. We need to hypothesize about these processes and develop protocols and specific tests for them.

Suppes (1973) and several authors in the Journal for Research in Mathematics Education have studied the step-by-step processes of learners engaged in mathematical tasks, such as addition and subtraction. I note also several interesting ATI studies on learners' abilities and verbal processes (summarized by Aiken, 1971) in the Journal for Research in Mathematics Education. Studies on these processes have great promise, I believe.

Brain research

Another important line of research is relevant to describing the process variables and individual differences involved in cognition and in the learning of mathematics. Recent research on the human brain indicates there is a differentiation of functions between its right and left hemispheres which may tell us something about how right or left hemispheres process information differently from each other. This research may also tell us how spatial and verbal processes are involved in learning.

In neuropsychological research with people whose brain hemispheres had been surgically deconnected by severing the corpus callosum, Sperry (1968) found that the verbal processes were controlled by the major brain hemisphere (the left one in typical right-handed people), while the spatial-imaginal processes were controlled by the minor hemisphere (the right one in typical right-handed people). His subjects correctly named printed words tachistoscopically presented for one-tenth of a second to the right visual field, which connects with the left hemisphere. These same subjects could not name nor were even aware of words tachistoscopically presented for one-tenth of a second through their left visual field to their right hemispheres. However, by feeling them with their left hands, which provide sensory feedback to the right-brain hemispheres, these same people correctly selected the objects that represented the words just presented to their right hemispheres. Even after correctly selecting the objects, the subjects still maintained that they had

never seen the words corresponding to the objects (Sperry, 1968, p. 725). Sperry's findings are consistent with the notion that there are two distinct neural processes which, although connected with each other in normal people, code spatial-imaginal stimuli and verbal stimuli differentially.

In a most interesting series of studies entitled "The Other Side of the Brain," Bogen and co-workers (Bogen, 1969a, 1969b; Bogen and Bogen, 1969; and Bogen et al., 1972) present the results of their research with "split-brain" subjects, i.e., people whose hemispheres have been severed by surgery. Bogen (1969b) describes the lateralization of the brain's functions as follows. Language, verbal and logical processes are primarily in the left hemisphere, while visual, spatial, Gestalt perceptual, and imaginal functions are primarily on the right side. He feels that the left hemisphere is specialized for propositional thought and the right hemisphere for appositional thought. He quotes Levy-Agresti and Sperry (1968) to suggest lateralization is as follows:

> The data indicate that the mute, minor hemisphere is specialized for Gestalt perception, being primarily a synthesist in dealing with information input. The speaking, major hemisphere, in contrast, seems to operate in a more logical, analytic, computer-like fashion [and] the findings suggest that a possible reason for cerebral lateralization in man is basic incompatibility of language functions on the one hand and synthetic perceptual functions on the other hand. (Bogen, 1969b, p. 149)

Figure 8 (based upon Bogen, 1969b, p. 150) summarizes some of the educationally relevant functions and processes which seem to be

lateralized in the brain, especially after the age of 5 years. Some of the categories within a column are repeated because of the overlapping comparisons made in the different studies whose findings are summarized in Figure 8.

Left Hemisphere	Right Hemisphere
propositional thought	appositional thought
language	spatial relations
verbal	perceptual
symbolic	visiospatial
temporal processing	part-whole processing (Gestalt perception)
logical or analytic	analogic or relational
propositional thought	visual imagery
linear	non-linear

Figure 8. Functions which seem to be lateralized in the brain.

From the above findings, it seems that our brains have at least two modes of learning, remembering and solving problems, as diagrammed above. If we have multiple modes of learning, we should use this valuable information in the individualization of instruction. Another possible meaning of the above research is that instruction should be introduced in our dominant or preferred mode and elaborated in our second mode. Again, I feel that research on the processing of information is a fruitful area for us to investigate.

Higher-order processes

If we are able to isolate higher-order processes, used in learning or in problem-solving strategies, can we teach students to improve

their use of them? The studies on imagery and verbal elaboration indicate that we can. The answer is also yes if you accept the following data of mine.

In 1967, I reported a study in the Journal of Educational Psychology on the teaching of replacement strategies and non-replacement strategies to young children (Wittrock, 1967). I used physical objects to represent the hypothesis to be tested. In the non-replacement strategy each card representing a hypothesis was turned face-down when it was eliminated from the set of testable hypotheses. In that study, the children learned and transferred simple problem-solving strategies to new problems, including ones where their cards and props no longer appeared.

I am convinced that higher-order strategies can be taught to children. I notice in the Journal for Research in Mathematics Education several studies on the teaching of strategies. This is an important area to pursue.

Motivation

Recent research in cognition also provides a new perspective regarding how reinforcement may operate upon students. I will mention two areas of research here: (1) achievement motivation and (2) delay-retention effect.

Weiner et al. (1972) and Weiner (1972) indicate that the effects of reinforcement depend upon which of the learner's attributes, e.g., his effort or his innate intellectual ability, he believes is being

reinforced. If the student infers that his effort at learning mathematics is responsible for his success, he is more likely to persevere with similar mathematical problems than if he infers that his success or failure is attributable to factors over which he has no control, such as his innate ability or lack of it. Weiner's cognitive interpretation of reinforcement has practical utility for teachers of mathematics. A student may erroneously assume his failure to learn is due to factors beyond his control, when additional effort would solve the mathematics problem and increase motivation to learn more advanced mathematics.

Delay-retention effect

For many years, it has been believed that reinforcers should be given (1) immediately, (2) discriminately, and (3) frequently, during acquisition of behavior. Teacher-education students studying educational psychology have been taught those "truths" about using reinforcement and feedback. However, recent data on the delay-retention effect (Sassenrath and Yonge, 1969) indicate that there is now reason to doubt the importance of immediate feedback. One reason for the doubt is that immediate feedback sometimes reduces learning when it is compared with delayed feedback, perhaps because it stops problem-solving and other cognitive processes.

The results presented in this paper are compatible with either S-R or cognitive principles, even though Skinner's logical positivistic approach does not emphasize that research in cognitive processes is

a productive way to study human behavior. Nevertheless, learners do construct relations from their instruction. Their attitudes do determine, in part at least, what the effects of the reinforcer are and what they will learn and remember. The research summarized above means to me that learning is a constructive, generative process.

III. Implications for the Teaching of Mathematics

From the research and theory developed above, I want to discuss implications for teaching and research in mathematics learning. In preparation for writing this implications section, I read many articles in issues of the Journal for Research in Mathematics Education. In addition, I read research articles and papers by many of you. I also read descriptions of the excellent mathematics curricula which are now being used throughout the public schools of the United States, such as the SMSG materials prepared under Dr. Begle's lead.

To my joy, I discovered that many researchers in mathematics education were delving into the same issues that I have discussed above. I found Suppes (1973) specifying the step-by-step processes involved in addition and subtraction; Atkinson (1973) considering the learner's detailed history in the design steps in an instructional program; ATI research using intellectual processing variables; and attempts to conceptualize mathematics learning from new perspectives (Scandura, 1971).

After reading some of the research in mathematics education, I tried to see where educational psychology and mathematics education

shared common interests: where each field seemed to have something to offer the other. First of all, it is clear that the two fields share many common interests, especially in the areas of cognition, concept learning, and problem-solving. I hope at next year's AERA annual meeting we can arrange a talk entitled "Recent Research in Mathematics Learning Applied to Educational Psychology".

The two implications for mathematics learning and teaching which I want to emphasize involve the organizational structures and processes discussed throughout this paper. In structural organization, mathematics education has made great advances in recent times, largely because of the carefully organized instructional materials prepared by SMSG and other curriculum projects. Their authors and researchers have emphasized hierarchical organization and meaning. The organization of mathematics around its most fundamental concepts and the teaching of them first to young children is a most important contribution to education. Research in ATI, brain processes, and individual differences is needed to determine if multiple structural organizations of mathematics will improve instruction.

In the area of processing of information, recent research in cognition has offered important ideas regarding mathematics learning. To summarize these ideas, I suggest that learning of mathematics be viewed as a generative process. This involves relating the structural organization to the learner's experience, and encouraging the learner to process the information. Let me give an example.

Before a mathematics teacher presents a theorem or equation with a statement such as "Let $x = 2y$," he should consider what his statements will cause the learner to do. What is the purpose of letting $x = 2y$? What is the learner to generage from the equation? How does the equation relate to the learner's experience; his previous mathematics? What verbal and spatial processes are involved in the new material?

After thinking about questions such as these, I conclude that it is important to study the step-by-step mathematical and higher-order cognitive processes involved in every phase of the learning of mathematics. It may seem trivial to focus on the details, but it is through the study of these specific processes that a deeper understanding of the learning of mathematics will develop. Some of the success of the recently developed curricular materials is due to their understanding of this point. A similar improvement can be made in the teaching of mathematics by conceptualizing the cognitive processes in it as generative ones.

Plato taught the slave boy to construct a proof. The boy's thought processes were engaged in the problem, even though Plato closely directed the boy's development of the proof. B. F. Skinner dislikes Plato's style of teaching for the very reason I like it most, that it was not directive enough. In other words, the instruction left some room for thought within an organized structure. It left something important for the learner to do; something to process, something to generate.

Accepting the notion that learning with understanding is a generative process requires explicit changes in our methods of teaching. When we walk into a classroom, we cannot assume that students are at the same level for storing and retrieving information, or that all students process information in the same manner. It is important to determine the individual's specific level of knowledge and his ways of processing information.

My research on human learning indicates that the ways the teacher introduces new material, the ways he relates it to the student's experiences, and the ways he stimulates the student to generate meanings are crucially important to learning. With proper attention to the introduction and sequencing of new material, we may need far fewer reinforced repetitions for learning to occur.

I do not accept Plato's ideas about nativism, about inheritance of abstractions. But I do accept his belief that the learner must actively construct meaning if he is to learn with understanding.

Today, after more than 2000 years of Aristotelian-dominated thought about human learning, Plato's ideas about teaching are still very much alive.

Let's put these ideas to use.

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